

# SPECTRAL ISOPERIMETRIC INEQUALITIES FOR $\delta$ -INTERACTIONS ON OPEN ARCS AND FOR THE ROBIN LAPLACIAN ON PLANES WITH SLITS

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**ABSTRACT.** We consider the problem of geometric optimization for the lowest eigenvalue of the two-dimensional Schrödinger operator with an attractive  $\delta$ -interaction supported on an open arc with two free endpoints. Under a constraint of fixed length of the arc, we prove that the maximizer is a line segment, the respective spectral isoperimetric inequality being strict. We also show that in the optimization problem for the same spectral quantity, but with the constraint of fixed endpoints, the optimizer is the line segment connecting them. Furthermore, we prove that a line segment is also the maximizer in the optimization problem for the lowest eigenvalue of the Robin Laplacian on a plane with a slit along an open arc of fixed length.

## 1. INTRODUCTION

The most classical spectral isoperimetric inequality states that among all planar domains of a given perimeter, the disc induces the lowest principal eigenvalue for the Dirichlet Laplacian. This statement follows from the famous Faber-Krahn inequality [F23, K24] via a simple scaling argument. In this paper we focus on related spectral isoperimetric properties for the principal eigenvalues of the two-dimensional Schrödinger operator with a  $\delta$ -interaction supported on an open arc and of the Robin Laplacian on a plane with a slit.

First, we discuss the results for Schrödinger operators with  $\delta$ -interactions. To this aim, let  $\Sigma \subset \mathbb{R}^2$  be any smooth compact closed or non-closed curve; *cf.* Section 2 for details. Given a real number  $\alpha > 0$ , consider the spectral problem for the self-adjoint operator  $H_{\delta,\alpha}^\Sigma$  corresponding via the first representation theorem to the closed, densely defined, symmetric, and semi-bounded quadratic form in  $L^2(\mathbb{R}^2)$

$$(1.1) \quad \mathfrak{h}_{\delta,\alpha}^\Sigma[u] := \|\nabla u\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)}^2 - \alpha \|u|_\Sigma\|_{L^2(\Sigma)}^2, \quad \text{dom } \mathfrak{h}_{\delta,\alpha}^\Sigma := H^1(\mathbb{R}^2);$$

here  $u|_\Sigma$  denotes the usual trace of  $u \in H^1(\mathbb{R}^2)$  onto  $\Sigma$ ; *cf.* [BEKS94, Sec. 2] and [BLL13, Sec. 3.2]. Typically,  $H_{\delta,\alpha}^\Sigma$  is called the Schrödinger operator with  $\delta$ -interaction of strength  $\alpha$  supported on  $\Sigma$ . The essential spectrum of  $H_{\delta,\alpha}^\Sigma$  coincides with the set  $[0, \infty)$  and its negative discrete spectrum is known to be non-empty; *cf.* Section 2. By  $\lambda_1^\alpha(\Sigma)$  we denote the lowest negative eigenvalue of  $H_{\delta,\alpha}^\Sigma$ . For the operator  $H_{\delta,\alpha}^\Sigma$  holds a

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2010 *Mathematics Subject Classification.* 35P15 (primary); 58J50 (secondary).

*Key words and phrases.* Singular Schrödinger operator,  $\delta$ -interaction on an open arc, Robin Laplacian on planes with slits, lowest eigenvalue, spectral isoperimetric inequality, Birman-Schwinger principle.

spectral isoperimetric inequality [E05, EHL06] analogous to the spectral isoperimetric inequality for the Dirichlet Laplacian mentioned above. To be more precise, it can be stated as follows

$$(1.2) \quad \max_{|\Sigma|=L} \lambda_1^\alpha(\Sigma) = \lambda_1^\alpha(C_{L/(2\pi)})$$

where the maximum is taken over all smooth loops of a given length  $L > 0$ . Here, we denote by  $|\Sigma|$  the length of  $\Sigma$  and  $C_{L/(2\pi)}$  is a circle of the radius  $R = L/(2\pi)$ . We remark that by [BFK<sup>+</sup>16] an analogue of (1.2) holds for  $\delta$ -interactions supported on curves in  $\mathbb{R}^3$  and according to the counterexample in [EF09] no direct analogue of (1.2) can hold in the space dimension  $d = 3$  for  $\delta$ -interactions supported on surfaces, except for special classes of surfaces [EL15].

In the last several years, the investigation of Schrödinger operators with singular interactions supported on non-closed curves and open surfaces became a topic of significant interest [DEKP16, EK16, EP14, ER16, JL16, MPS16a, MPS16b]. In this paper, we obtain a counterpart of (1.2) for two-dimensional Schrödinger operators with  $\delta$ -interactions supported on open arcs with the optimizer being a line segment. The respective statement is precisely formulated below.

**Theorem 1.1.** *For all  $\alpha > 0$ , holds*

$$(1.3) \quad \max_{|\Sigma|=L} \lambda_1^\alpha(\Sigma) = \lambda_1^\alpha(\Gamma_L)$$

where the maximum is taken over all smooth open arcs of a given length  $L > 0$  and  $\Gamma_L$  denotes a line segment of length  $L$ . The equality in (1.3) is possible if, and only if,  $\Sigma$  and  $\Gamma_L$  are congruent.

Our method of the proof of Theorem 1.1 relies on the Birman-Schwinger principle for  $H_{\delta,\alpha}^\Sigma$  and on the trick proposed in [E05, EHL06] and further applied and developed in [BFK<sup>+</sup>16, EL15]. The main geometric ingredient in the proof of Theorem 1.1 is that the line segment is the shortest path connecting two fixed endpoints. However, this simple geometric fact pops up in a somewhat unusual context. We point out that a result similar to (1.3) can also be proven under the constraint of fixed endpoints while the length of the arc varies; see the discussion in Subsection 5.1. In fact, the latter claim is a consequence of Theorem 1.1 and of the ordering between the eigenvalues of  $H_{\delta,\alpha}^\Gamma$  and of  $H_{\delta,\alpha}^\Lambda$  under the inclusion  $\Gamma \subset \Lambda$ .

Second, we describe the results for the Robin Laplacian on a plane with a slit. Let  $\Sigma \subset \mathbb{R}^2$  be a smooth compact non-closed curve as above. For a real number  $\alpha > 0$ , consider the spectral problem for the self-adjoint Robin Laplacian  $H_{R,\alpha}^\Sigma$  on  $\mathbb{R}^2 \setminus \Sigma$  which corresponds via the first representation theorem to the closed, densely defined, symmetric, and semi-bounded quadratic form in  $L^2(\mathbb{R}^2)$

$$(1.4) \quad \begin{aligned} \mathfrak{h}_{R,\alpha}^\Sigma[u] &:= \|\nabla u\|_{L^2(\mathbb{R}^2;\mathbb{C}^2)}^2 - \alpha \left( \|u|_{\Sigma_+}\|_{L^2(\Sigma)}^2 + \|u|_{\Sigma_-}\|_{L^2(\Sigma)}^2 \right), \\ \text{dom } \mathfrak{h}_{R,\alpha}^\Sigma &:= H^1(\mathbb{R}^2 \setminus \Sigma); \end{aligned}$$

here  $u|_{\Sigma_\pm}$  denote the traces of  $u \in H^1(\mathbb{R}^2 \setminus \Sigma)$  onto two faces of  $\Sigma$ . It is known that the essential spectrum of  $H_{R,\alpha}^\Sigma$  coincides with  $[0, \infty)$ . By a simple variational argument

one gets that the negative discrete spectrum of  $H_{R,\alpha}^\Sigma$  is also non-empty. We denote by  $\mu_1^\alpha(\Sigma)$  the lowest negative eigenvalue of  $H_{R,\alpha}^\Sigma$  and obtain a claim for the Robin Laplacian on  $\mathbb{R}^2 \setminus \Sigma$  analogous to Theorem 1.1.

**Theorem 1.2.** *For all  $\alpha > 0$ , holds*

$$(1.5) \quad \max_{|\Sigma|=L} \mu_1^\alpha(\Sigma) = \mu_1^\alpha(\Gamma_L)$$

where the maximum is taken over all smooth open arcs of a given length  $L > 0$  and  $\Gamma_L$  denotes a line segment of length  $L$ . The equality in (1.5) is possible if, and only if,  $\Sigma$  and  $\Gamma_L$  are congruent.

We achieve the proof of Theorem 1.2 via a combination of Theorem 1.1 and of a trick based on the symmetry and on the ordering between the forms  $\mathfrak{h}_{R,\alpha}^\Sigma$  and  $\mathfrak{h}_{\delta,2\alpha}^\Sigma$ . It is worth mentioning that, unlike in our setting, the isoperimetric property (1.2) for loops does not imply any claim of such a kind for Robin Laplacians on planar domains with compact boundaries. For Robin Laplacians on bounded domains, different methods are developed for repulsive [B86, D06] and attractive [AFK16, FK15] boundary conditions. The method for attractive boundary conditions is further generalized in [KL16] to exterior domains.

The organisation of this paper is as follows. In Section 2 we recall basic known spectral properties of  $H_{\delta,\alpha}^\Sigma$  that are needed in this paper. Section 3 is devoted to the Birman-Schwinger principle for  $H_{\delta,\alpha}^\Sigma$  and its consequences. Theorem 1.1 is proven in Section 4. The paper is concluded by Section 5 with applications of Theorem 1.1. Namely, in Subsection 5.1 we discuss the optimization of  $\lambda_1^\alpha(\Sigma)$  under the constraint of fixed endpoints for  $\Sigma$  and in Subsection 5.2 we prove Theorem 1.2 concerning the optimization of the lowest eigenvalue of the Robin Laplacian on a plane with a slit.

## 2. THE SPECTRAL PROBLEM FOR $\delta$ -INTERACTIONS SUPPORTED ON OPEN ARCS

Throughout this section,  $\Sigma$  is an arbitrary curve of a finite length in  $\mathbb{R}^2$  with two free endpoints. For simplicity, we assume that  $\Sigma$  is smooth (*i.e.*  $C^\infty$ -smooth), but less regularity is evidently needed for the majority of the results to hold. We emphasize that by saying that  $\Sigma$  is smooth we implicitly understand that it can be continued up to the boundary of a  $C^\infty$ -smooth bounded simply connected domain. In particular,  $\Sigma$  has no self-intersections and no increasing oscillations at the endpoints. At the same time,  $\alpha$  is an arbitrary positive real number.

We are interested in the spectral properties of the self-adjoint operator  $H_{\delta,\alpha}^\Sigma$  in  $L^2(\mathbb{R}^2)$  introduced via the first representation theorem [K, Thm. VI 2.1] through the closed, densely defined, symmetric and semi-bounded quadratic form  $\mathfrak{h}_{\delta,\alpha}^\Sigma$  in (1.1); see [BEKS94, Sec. 2] and also [BLL13]. Let  $\tilde{\Sigma}$  be a continuation of  $\Sigma$  up to the boundary of a bounded smooth domain  $\Omega_+ \subset \mathbb{R}^2$  and let  $\Omega_- := \mathbb{R}^2 \setminus \overline{\Omega_+}$  be the complement of  $\Omega_+$ . For any  $u \in L^2(\mathbb{R}^2)$  we introduce the notation  $u_\pm := u|_{\Omega_\pm}$ . Then the operator domain of  $H_{\delta,\alpha}^\Sigma$  consists of functions  $u \in H^1(\mathbb{R}^2)$  which satisfy  $\Delta u_\pm \in L^2(\Omega_\pm)$  in the distributional

sense and  $\delta$ -type boundary conditions

$$(2.1) \quad \partial_{\nu_+} u_+|_{\tilde{\Sigma}} + \partial_{\nu_-} u_-|_{\tilde{\Sigma}} = \alpha \chi_{\Sigma} u|_{\tilde{\Sigma}}$$

on  $\tilde{\Sigma}$  in the sense of traces, where  $\chi_{\Sigma}: \tilde{\Sigma} \rightarrow \tilde{\Sigma}$  is the characteristic function of  $\Sigma$  in  $L^2(\tilde{\Sigma})$  and where  $\partial_{\nu_{\pm}} u_{\pm}|_{\tilde{\Sigma}}$  denote the traces of normal derivatives of  $u_{\pm}$  onto  $\tilde{\Sigma}$  with the normal vectors pointing outwards  $\Omega_{\pm}$ . Moreover, for any  $u \in \text{dom } H_{\delta, \alpha}^{\Sigma}$  we have  $H_{\delta, \alpha}^{\Sigma} u = (-\Delta u_+) \oplus (-\Delta u_-)$ . The reader may consult with [MPS16a, Cor. 6.21] and [BLL13, Sec. 3.2] for a more precise description of  $\text{dom } H_{\delta, \alpha}^{\Sigma}$ .

The operator  $H_{\delta, \alpha}^{\Sigma}$  possesses a non-empty essential spectrum. Namely, we have the following statement.

**Proposition 2.1.** *For all  $\alpha > 0$  holds  $\sigma_{\text{ess}}(H_{\delta, \alpha}^{\Sigma}) = [0, \infty)$ .*

The claim of this proposition is expected because the essential spectrum of the Laplacian in the whole space  $\mathbb{R}^2$  equals  $[0, \infty)$  and introducing a  $\delta$ -interaction supported on  $\Sigma$  leads to a compact perturbation in the sense of resolvent differences. The proofs of Proposition 2.1 can be found in [BEKS94, Thm. 3.1] and also in [BLL13, Thm. 4.3].

Various properties of the discrete spectrum of  $H_{\delta, \alpha}^{\Sigma}$  are investigated in [BLL13, EP14, KL14]. For our purposes we only require the following statement.

**Proposition 2.2.** *For all  $\alpha > 0$  holds  $1 \leq \#\sigma_d(H_{\delta, \alpha}^{\Sigma}) < \infty$ <sup>1</sup>.*

For a proof of  $1 \leq \#\sigma_d(H_{\delta, \alpha}^{\Sigma})$  see [KL14, Thm. 3.1]. Non-emptiness of  $\sigma_d(H_{\delta, \alpha}^{\Sigma})$  can alternatively be shown by the min-max principle with the aid of the family of test functions having the same structure as in the proof of [KL16, Prop. 2]. A simple proof of  $\#\sigma_d(H_{\delta, \alpha}^{\Sigma}) < \infty$  can be found in [BLL13, Thm. 3.14]. Finiteness of the discrete spectrum for  $H_{\delta, \alpha}^{\Sigma}$  may also be derived from the spectral estimate in [BEKS94, Thm. 4.2 (iii)].

Finally, we obtain fundamental properties of the lowest eigenvalue  $\lambda_1^{\alpha}(\Sigma)$  for  $H_{\delta, \alpha}^{\Sigma}$  and of the corresponding eigenspace.

**Proposition 2.3.** *For all  $\alpha > 0$ , the lowest eigenvalue  $\lambda_1^{\alpha}(\Sigma) < 0$  of  $H_{\delta, \alpha}^{\Sigma}$  is simple and the corresponding eigenfunction can be chosen to be non-negative in  $\mathbb{R}^2$ .*

*Proof.* The argument follows the same strategy as the proof of [GT, Thm. 8.38]. Denote  $\lambda := \lambda_1^{\alpha}(\Sigma) < 0$  and let  $u = u_+ \oplus u_- \in \ker(H_{\delta, \alpha}^{\Sigma} - \lambda)$ . By standard elliptic regularity we get  $u_{\pm} \in H_{\text{loc}}^2(\Omega_{\pm})$ . Without loss of generality we can assume that  $u$  is real-valued and that  $\|u\|_{L^2(\mathbb{R}^2)} = 1$ . Clearly, we have  $|u| \in H^1(\mathbb{R}^2)$ ,  $\||u|\|_{L^2(\mathbb{R}^2)} = 1$ , and, moreover,  $\mathfrak{h}_{\delta, \alpha}^{\Sigma}[|u|] = \mathfrak{h}_{\delta, \alpha}^{\Sigma}[u]$ . The condition that  $|u|$  is a minimizer for the quadratic form  $\mathfrak{h}_{\delta, \alpha}^{\Sigma}$  implies a characterization of  $|u|$  through an Euler-Lagrange-type equation

$$(2.2) \quad \mathfrak{h}_{\delta, \alpha}^{\Sigma}[|u|, \phi] = \lambda(|u|, \phi)_{L^2(\mathbb{R}^2)}, \quad \forall \phi \in H^1(\mathbb{R}^2),$$

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<sup>1</sup>We denote by  $\#\sigma_d(T)$  the number of discrete eigenvalues with multiplicities taken into account for a self-adjoint operator  $T$ .

which is equivalent to the variational characterization of an eigenfunction for  $H_{\delta,\alpha}^\Sigma$  corresponding to the eigenvalue  $\lambda$ . Thus, we have  $|u| \in \ker(H_{\delta,\alpha}^\Sigma - \lambda)$ . In particular, we have shown that  $-\Delta|u_\pm| = \lambda|u_\pm|$  holds on  $\Omega_\pm$  in the distributional sense. Thus, by elliptic regularity we also get  $|u_\pm| \in H_{\text{loc}}^2(\Omega_\pm)$ .

Clearly,  $u_+ = 0$  and  $u_- = 0$  do not hold simultaneously taking into account that  $\|u\|_{L^2(\mathbb{R}^2)} = 1$ . If either  $u_+ = 0$  or  $u_- = 0$ , then  $u \in H^1(\mathbb{R}^2)$  implies that  $u$  satisfies Dirichlet boundary conditions on  $\tilde{\Sigma}$  and we get a contradiction to non-negativity of the Dirichlet Laplacians on  $\Omega_\pm$ .

Furthermore, Harnack's inequality [GT, Cor. 8.21] yields that  $|u_\pm|$  are pointwise positive in  $\Omega_\pm$ . Thus, standard properties of  $H^1$ -functions imply that  $u_\pm$  are sign-definite pointwise non-vanishing functions in  $\Omega_\pm$ . It remains to exclude the case when  $u_\pm$  are of different signs. Indeed, if it were the case, then in view of  $u \in H^1(\mathbb{R}^2)$  we would get  $u|_{\tilde{\Sigma}} = 0$ . Thus,  $u_\pm$  would be simultaneously eigenfunctions of Dirichlet Laplacians on  $\Omega_\pm$  corresponding to a negative eigenvalue  $\lambda < 0$ , which is a contradiction. Hence, we obtain that either  $u = |u|$  or  $u = -|u|$ .

This argument shows that any function in  $\ker(H_{\delta,\alpha}^\Sigma - \lambda)$  is pointwise positive in  $\mathbb{R}^2 \setminus \tilde{\Sigma}$  and non-negative in  $\mathbb{R}^2$  (up to multiplication by a constant factor). Hence, it is impossible that  $\ker(H_{\delta,\alpha}^\Sigma - \lambda)$  contains two linearly independent functions that are orthogonal to each other. Thus, we obtain that the linear subspace  $\ker(H_{\delta,\alpha}^\Sigma - \lambda)$  of  $\text{dom } H_{\delta,\alpha}^\Sigma$  is one-dimensional.  $\square$

Summarizing, the essential spectrum of  $H_{\delta,\alpha}^\Sigma$  equals the interval  $[0, \infty)$  and there is at least one discrete eigenvalue below 0. In particular, the lowest point  $\lambda_1^\alpha(\Sigma)$  in the spectrum is always a simple negative discrete eigenvalue and the corresponding eigenfunction can be selected to be non-negative in  $\mathbb{R}^2$ .

### 3. BIRMAN-SCHWINGER PRINCIPLE

In this section we formulate a Birman-Schwinger-type principle for the operator  $H_{\delta,\alpha}^\Sigma$  and derive a related characterization of its lowest eigenvalue  $\lambda_1^\alpha(\Sigma)$ .

First, we parametrize the curve  $\Sigma$  by the unit-speed mapping  $\Sigma: \mathcal{I} \rightarrow \mathbb{R}^2$  with  $\mathcal{I} := [0, L]$ ; *i.e.*  $|\dot{\Sigma}(s)| = 1$  for all  $s \in \mathcal{I}$ . Clearly, the Hilbert spaces  $L^2(\Sigma)$  and  $L^2(\mathcal{I})$  can be identified. Second, we define a weakly singular integral operator  $Q^\Sigma(\kappa): L^2(\mathcal{I}) \rightarrow L^2(\mathcal{I})$  for  $\kappa > 0$  by

$$(3.1) \quad (Q^\Sigma(\kappa)\psi)(s) := \frac{1}{2\pi} \int_0^L K_0(\kappa|\Sigma(s) - \Sigma(s')|) \psi(s') ds',$$

where  $K_0(\cdot)$  is the modified Bessel function of the second kind and of the order  $\nu = 0$ ; *cf.* [AS64, §9.6]. In the next proposition we state basic properties of this integral operator.

**Proposition 3.1.** *The operator  $Q^\Sigma(\kappa)$  in (3.1) is self-adjoint, compact, and non-negative for all  $\kappa > 0$ .*

*Proof.* Compactness of  $Q^\Sigma(\kappa)$  is proven in [BEKS94, Lem. 3.2]. Self-adjointness and non-negativity of  $Q^\Sigma(\kappa)$  follow from abstract results in [B95].  $\square$

Now we have all the tools to formulate a Birman-Schwinger-type condition for  $H_{\delta,\alpha}^\Sigma$ .

**Theorem 3.2.** *Let the self-adjoint operator  $H_{\delta,\alpha}^\Sigma$  in  $L^2(\mathbb{R}^2)$  represent the quadratic form in (1.1) and let the operator-valued function  $\mathbb{R}_+ \ni \kappa \mapsto Q^\Sigma(\kappa)$  be as in (3.1). Then the following claims hold.*

- (i)  $\dim \ker(H_{\delta,\alpha}^\Sigma + \kappa^2) = \dim \ker(I - \alpha Q^\Sigma(\kappa))$  for all  $\kappa > 0$ .
- (ii) The mapping  $u \mapsto u|_\Sigma$  is a bijection between  $\ker(H_{\delta,\alpha}^\Sigma + \kappa^2)$  and  $\ker(I - \alpha Q^\Sigma(\kappa))$ .

*Proof.* For the proof of (i) see [BEKS94, Lem. 2.3 (iv)] and also [BLL13, Thm. 3.5 (iii)]. The claim of (ii) is a consequence of the abstract statement in [B95, Lem. 1].  $\square$

We conclude this section by two corollaries of Theorem 3.2.

**Corollary 3.3.** *Let the assumptions be as in Theorem 3.2 and let  $\kappa > 0$  be such that  $\lambda_1^\alpha(\Sigma) = -\kappa^2$ . Then the following claims hold.*

- (i)  $\dim \ker(I - \alpha Q^\Sigma(\kappa)) = 1$ .
- (ii)  $\ker(I - \alpha Q^\Sigma(\kappa)) = \text{span}\{\psi\}$  where  $\psi \in L^2(\mathcal{I})$  is a positive function.

*Proof.* Recall that by Proposition 2.3 the lowest eigenvalue  $\lambda = \lambda_1^\alpha(\Sigma)$  of  $H_{\delta,\alpha}^\Sigma$  is simple. Hence, the claim of (i) immediately follows from Theorem 3.2 (i).

Denote by  $\psi \in L^2(\mathcal{I})$  the trace on  $\Sigma$  of the eigenfunction of  $H_{\delta,\alpha}^\Sigma$  corresponding to its lowest eigenvalue  $\lambda$ . According to Theorem 3.2 (ii) we have  $\ker(I - \alpha Q^\Sigma(\kappa)) = \text{span}\{\psi\}$ . Furthermore, recall that by Proposition 2.3 the eigenfunction of  $H_{\delta,\alpha}^\Sigma$  corresponding to the lowest eigenvalue  $\lambda$  can be chosen to be non-negative in  $\mathbb{R}^2$ . Clearly, the trace on  $\Sigma$  of an  $H^1$ -function, that is non-negative in  $\mathbb{R}^2$ , is non-negative as well. Thus, we can select the function  $\psi$  to be non-negative.

Finally, the identity  $\psi = \alpha Q^\Sigma(\kappa)\psi$ , non-negativity of  $\psi$ , and strict positivity of the integral kernel of  $Q^\Sigma(\kappa)$  in (3.1) imply that  $\psi$  is, in fact, positive.  $\square$

Now we provide the second consequence of Theorem 3.2.

**Corollary 3.4.** *Let the assumptions be as in Theorem 3.2. Then the following claims hold.*

- (i)  $\sup \sigma(\alpha Q^\Sigma(\kappa)) \geq 1$  if, and only if,  $\lambda_1^\alpha(\Sigma) \leq -\kappa^2$ .
- (ii)  $\sup \sigma(\alpha Q^\Sigma(\kappa)) = 1$  if, and only if,  $\lambda_1^\alpha(\Sigma) = -\kappa^2$ .

*Proof.* In the proof it will be convenient to use the following shorthand notations:

$$(3.2) \quad F_\alpha(\kappa) := \sup \sigma(\alpha Q^\Sigma(\kappa)) \quad \text{and} \quad G(\alpha) := \lambda_1^\alpha(\Sigma).$$

First, we recall that the function  $\mathbb{R}_+ \ni \kappa \mapsto F_\alpha(\kappa)$  is continuous [BEKS94, Lem 3.2] and strictly decaying (cf. [BLL13, Prop. 3.2] and [BLLR15, Lem. 2.3 (i)]) and that



$F_\alpha(\kappa) \rightarrow 0+$  as  $\kappa \rightarrow +\infty$  (cf. [GS15, Thm. 3.1]). Second, recall that the function  $\mathbb{R}_+ \ni \alpha \mapsto G(\alpha)$  is also continuous and strictly decaying, and that  $G(\alpha) \rightarrow -\infty$  as  $\alpha \rightarrow +\infty$  (see e.g. [L14, Prop. 2.9]). Now we pass to the proofs of the claims.

(i)  $F_\alpha(\kappa) \geq 1$  implies that for some  $\nu \geq \kappa$  holds  $F_\alpha(\nu) = 1$ . Therefore, by Proposition 3.1 and Theorem 3.2 (i) we have  $-\nu^2 \in \sigma_d(H_{\delta,\alpha}^\Sigma)$  and, in particular,  $G(\alpha) \leq -\nu^2 \leq -\kappa^2$ .

Suppose now that  $G(\alpha) \leq -\kappa^2$ . Then there exists  $\nu \geq \kappa$  such that  $-\nu^2 \in \sigma_d(H_{\delta,\alpha}^\Sigma)$ . By Proposition 3.1 and Theorem 3.2 (i) we get  $1 \in \sigma_d(\alpha Q^\Sigma(\nu))$  and thus  $F_\alpha(\nu) \geq 1$ . Finally, we have  $F_\alpha(\kappa) \geq F_\alpha(\nu) \geq 1$ .

(ii)  $F_\alpha(\kappa) = 1$  implies that  $G(\alpha) \leq -\kappa^2$  by (i). On the other hand, if  $G(\alpha) < -\kappa^2$  then for some  $\beta < \alpha$  holds  $G(\beta) = -\kappa^2$  and hence  $F_\alpha(\kappa) > F_\beta(\kappa) \geq 1$  which is a contradiction.

$G(\alpha) = -\kappa^2$  implies that  $F_\alpha(\kappa) \geq 1$  again by (i). On the other hand, if  $F_\alpha(\kappa) > 1$  then there exists  $\beta < \alpha$  such that  $F_\beta(\kappa) = 1$ . Thus, we have  $G(\alpha) < G(\beta) \leq -\kappa^2$  which is also a contradiction.  $\square$

#### 4. PROOF OF THEOREM 1.1

Now we are in a position to establish Theorem 1.1. Throughout this section,  $\Sigma \subset \mathbb{R}^2$  is a compact  $C^\infty$ -smooth curve of length  $L > 0$  with two free endpoints which is parametrized by the unit-speed mapping  $\Sigma: \mathcal{I} \rightarrow \mathbb{R}^2$  with  $\mathcal{I} = [0, L]$  and  $\Gamma = \Gamma_L \subset \mathbb{R}^2$  is a line segment having the same length  $L$  which is parametrized by the unit-speed mapping  $\Gamma: \mathcal{I} \rightarrow \mathbb{R}^2$ . In addition, assume that  $\Sigma$  is not congruent to  $\Gamma$ .

Recall that  $\lambda_1^\alpha(\Sigma)$  and  $\lambda_1^\alpha(\Gamma)$  denote the lowest eigenvalues of  $H_{\delta,\alpha}^\Sigma$  and of  $H_{\delta,\alpha}^\Gamma$ , respectively. Furthermore, we fix  $\kappa > 0$  such that  $\lambda_1^\alpha(\Gamma) = -\kappa^2$ . By Corollary 3.3 (ii) we have  $\ker(I - \alpha Q^\Gamma(\kappa)) = \text{span}\{\psi\}$  where  $\psi \in L^2(\mathcal{I})$  is a positive function. Without loss of generality we assume that  $\|\psi\|_{L^2(\mathcal{I})} = 1$ . Observe that by Corollary 3.4 (ii) holds  $\sup \sigma(\alpha Q^\Gamma(\kappa)) = 1$ .

Note that for any  $s, s' \in \mathcal{I}$  we have

$$(4.1) \quad |\Sigma(s) - \Sigma(s')| \leq |\Gamma(s) - \Gamma(s')|.$$

Since  $\Sigma$  is not congruent to  $\Gamma$ , for simple geometric reasons there is a subset  $\mathcal{S} \subset \mathcal{I}^2$  having positive Lebesgue measure such that

$$(4.2) \quad |\Sigma(s) - \Sigma(s')| < |\Gamma(s) - \Gamma(s')|, \quad \forall (s, s') \in \mathcal{S}.$$

Using (4.1), (4.2), positivity of  $\psi$ , strict decay of  $K_0(\cdot)$ , and the min-max principle we obtain

$$\begin{aligned} \sup \sigma(\alpha Q^\Sigma(\kappa)) &\geq \frac{\alpha}{2\pi} \int_0^L \int_0^L K_0(\kappa |\Sigma(s) - \Sigma(s')|) \psi(s) \psi(s') \, ds \, ds' \\ &> \frac{\alpha}{2\pi} \int_0^L \int_0^L K_0(\kappa |\Gamma(s) - \Gamma(s')|) \psi(s) \psi(s') \, ds \, ds' \\ &= \sup \sigma(\alpha Q^\Gamma(\kappa)) = 1. \end{aligned}$$

Hence, by Corollary 3.4 we get

$$\lambda_1^\alpha(\Sigma) < -\kappa^2 = \lambda_1^\alpha(\Gamma).$$

Thus, the proof of the theorem is complete.  $\square$

## 5. CONSEQUENCES OF THEOREM 1.1

Let us conclude the paper by two consequences of Theorem 1.1, which are of certain independent interest.

**5.1. Fixed endpoints.** In this subsection we consider a related optimization problem for the lowest eigenvalue of  $H_{\delta,\alpha}^\Sigma$  under the constraint of fixed endpoints. We emphasize that no additional restrictions on the length of  $\Sigma$  are imposed.

**Proposition 5.1.** *For all  $\alpha > 0$ , holds*

$$(5.1) \quad \max_{\partial\Sigma=\{P,Q\}} \lambda_1^\alpha(\Sigma) = \lambda_1^\alpha(\Gamma_L)$$

where the maximum is taken over all smooth open arcs  $\Sigma$  that connect two given points  $P, Q \in \mathbb{R}^2$ ,  $P \neq Q$ , and  $\Gamma_L$  is a line segment of length  $L = |P - Q|$  where  $|P - Q|$  is the Euclidean distance between the points  $P$  and  $Q$ . The equality in (5.1) is possible if, and only if,  $\Sigma$  is the line segment that connects the points  $P$  and  $Q$ .

*Proof.* Let  $\Sigma \subset \mathbb{R}^2$  be any smooth open arc connecting the points  $P$  and  $Q$  which does not coincide with the line segment between them.

First, applying Theorem 1.1, we obtain

$$(5.2) \quad \lambda_1^\alpha(\Sigma) < \lambda_1^\alpha(\Lambda),$$

where  $\Lambda = \Lambda_{|\Sigma|}$  is a line segment of length  $|\Sigma|$ .

Second, observe that the following simple geometric inequality  $|\Sigma| > L = |P - Q|$  holds. Furthermore, let  $\Gamma = \Gamma_L$  be a line segment of length  $L$ . Without loss of generality we assume that  $\Gamma \subset \Lambda$ . Using the min-max principle and the form ordering  $\mathfrak{h}_{\delta,\alpha}^\Lambda \prec \mathfrak{h}_{\delta,\alpha}^\Gamma$  we arrive at

$$(5.3) \quad \lambda_1^\alpha(\Lambda) \leq \lambda_1^\alpha(\Gamma).$$

The claim of the proposition follows directly from (5.2) and (5.3).  $\square$

*Remark 5.2.* The proof of Proposition 5.1 indicates a way to obtain a quantified version of the spectral isoperimetric inequality under the constraint of fixed endpoints in the spirit of [BP12]. To this aim it suffices to obtain in the last step of the proof a positive lower bound on the difference  $\lambda_1^\alpha(\Gamma) - \lambda_1^\alpha(\Lambda)$  in terms of  $\alpha$ ,  $|\Gamma|$ , and  $|\Lambda|$ .



**5.2. The Robin Laplacian on  $\mathbb{R}^2 \setminus \Sigma$ .** The aim of this subsection is to prove Theorem 1.2 on the isoperimetric inequality for the Robin Laplacian  $H_{R,\alpha}^\Sigma$  on a plane with a slit  $\mathbb{R}^2 \setminus \Sigma$ . We recall that the self-adjoint operator  $H_{R,\alpha}^\Sigma$  in  $L^2(\mathbb{R}^2)$  is introduced via the first representation theorem through the closed, densely defined, symmetric and semi-bounded quadratic form  $\mathfrak{h}_{R,\alpha}^\Sigma$  in (1.4); cf. [ER16, Lem. 2.2] and also [MPS16a].

It is worth to mention already in the beginning of this subsection that for any  $\alpha > 0$  the form ordering  $\mathfrak{h}_{R,\alpha}^\Sigma \prec \mathfrak{h}_{\delta,2\alpha}^\Sigma$  holds thanks to the inclusion  $H^1(\mathbb{R}^2) \subset H^1(\mathbb{R}^2 \setminus \Sigma)$  and to the identity  $\mathfrak{h}_{R,\alpha}^\Sigma[u] = \mathfrak{h}_{\delta,2\alpha}^\Sigma[u]$ , which is satisfied for all  $u \in H^1(\mathbb{R}^2)$ .

First, we provide the following statement on the qualitative spectral properties of  $H_{R,\alpha}^\Sigma$ .

**Proposition 5.3.** *For all  $\alpha > 0$  holds  $\sigma_{\text{ess}}(H_{R,\alpha}^\Sigma) = [0, \infty)$  and  $1 \leq \#\sigma_d(H_{R,\alpha}^\Sigma) < \infty$ .*

*Proof.* The statements  $\sigma_{\text{ess}}(H_{R,\alpha}^\Sigma) = [0, \infty)$  and  $\#\sigma_d(H_{R,\alpha}^\Sigma) < \infty$  are special cases of [ER16, Thm. 3.1]. In view of the ordering  $\mathfrak{h}_{R,\alpha}^\Sigma \prec \mathfrak{h}_{\delta,2\alpha}^\Sigma$ , the property  $1 \leq \#\sigma_d(H_{R,\alpha}^\Sigma)$  follows from Proposition 2.2 and the min-max principle.  $\square$

Now we have all the tools to provide a proof of Theorem 1.2.

*Proof of Theorem 1.2.* Let  $\Sigma \subset \mathbb{R}^2$  be a compact  $C^\infty$ -smooth curve of length  $L > 0$  with two free endpoints and let  $\Gamma = \Gamma_L \subset \mathbb{R}^2$  be a line segment of the same length. For convenience we introduce Cartesian coordinates  $(x, y)$  on  $\mathbb{R}^2$ . Without loss of generality we assume that the line segment  $\Gamma$  is lying on the  $x$ -axis.

Recall that  $\mu_1^\alpha(\Sigma)$  and  $\mu_1^\alpha(\Gamma)$  denote the lowest eigenvalues of  $H_{R,\alpha}^\Sigma$  and of  $H_{R,\alpha}^\Gamma$ , respectively. First, we observe that in view of the ordering  $\mathfrak{h}_{R,\alpha}^\Sigma \prec \mathfrak{h}_{\delta,2\alpha}^\Sigma$  we have

$$(5.4) \quad \mu_1^\alpha(\Sigma) \leq \lambda_1^{2\alpha}(\Sigma).$$

Furthermore, we consider the subspaces  $L_{\text{even}}^2(\mathbb{R}^2)$  and  $L_{\text{odd}}^2(\mathbb{R}^2)$  of  $L^2(\mathbb{R}^2)$ , which consist, respectively, of even and odd functions in the  $y$ -variable. Both the subspaces  $L_{\text{even}}^2(\mathbb{R}^2)$  and  $L_{\text{odd}}^2(\mathbb{R}^2)$  can be identified with  $L^2(\mathbb{R}_+^2)$  via natural unitary transforms. With respect to the decomposition  $L^2(\mathbb{R}^2) = L_{\text{even}}^2(\mathbb{R}^2) \oplus L_{\text{odd}}^2(\mathbb{R}^2)$  and in view of the above identifications the operators  $H_{R,\alpha}^\Gamma$  and  $H_{\delta,2\alpha}^\Gamma$  can be decomposed into orthogonal sums

$$(5.5) \quad H_{R,\alpha}^\Gamma = A_\alpha \oplus B_\alpha \quad \text{and} \quad H_{\delta,2\alpha}^\Gamma = A_\alpha \oplus C,$$

where the self-adjoint operators  $A_\alpha$ ,  $B_\alpha$ , and  $C$  acting in  $L^2(\mathbb{R}_+^2)$  are introduced via the first representation theorem through closed, densely defined, symmetric, and semi-bounded quadratic forms

$$\begin{aligned} \mathfrak{a}_\alpha[u] &= \|\nabla u\|_{L^2(\mathbb{R}_+^2; \mathbb{C}^2)}^2 - \alpha \|u|_\Gamma\|_{L^2(\Gamma)}^2, & \text{dom } \mathfrak{a}_\alpha &= H^1(\mathbb{R}_+^2), \\ \mathfrak{b}_\alpha[u] &= \|\nabla u\|_{L^2(\mathbb{R}_+^2; \mathbb{C}^2)}^2 - \alpha \|u|_\Gamma\|_{L^2(\Gamma)}^2, & \text{dom } \mathfrak{b}_\alpha &= \{u \in H^1(\mathbb{R}_+^2) : u|_{\partial\mathbb{R}_+^2 \setminus \Gamma} = 0\}, \\ \mathfrak{c}[u] &= \|\nabla u\|_{L^2(\mathbb{R}_+^2; \mathbb{C}^2)}^2, & \text{dom } \mathfrak{c} &= H_0^1(\mathbb{R}_+^2). \end{aligned}$$

It can be easily seen that the operator  $C$  is non-negative. In view of the ordering  $\mathfrak{a}_\alpha \prec \mathfrak{b}_\alpha$ , the min-max principle implies that  $\inf \sigma(A_\alpha) \leq \inf \sigma(B_\alpha)$ . Thus, using

decompositions (5.5) we end up with

$$(5.6) \quad \mu_1^\alpha(\Gamma) = \inf \sigma(A_\alpha) = \lambda_1^{2\alpha}(\Gamma).$$

The claim immediately follows from (5.4), (5.6), and Theorem 1.1.  $\square$

## ACKNOWLEDGEMENTS

The author was supported by the grant No. 14-06818S of the Czech Science Foundation (GAČR). He is very grateful to Pavel Exner and David Krejčířík for fruitful discussions on the subject.

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